Introduction to Signal Dependent Transform

an insertion into Rolands Shavelis presentation

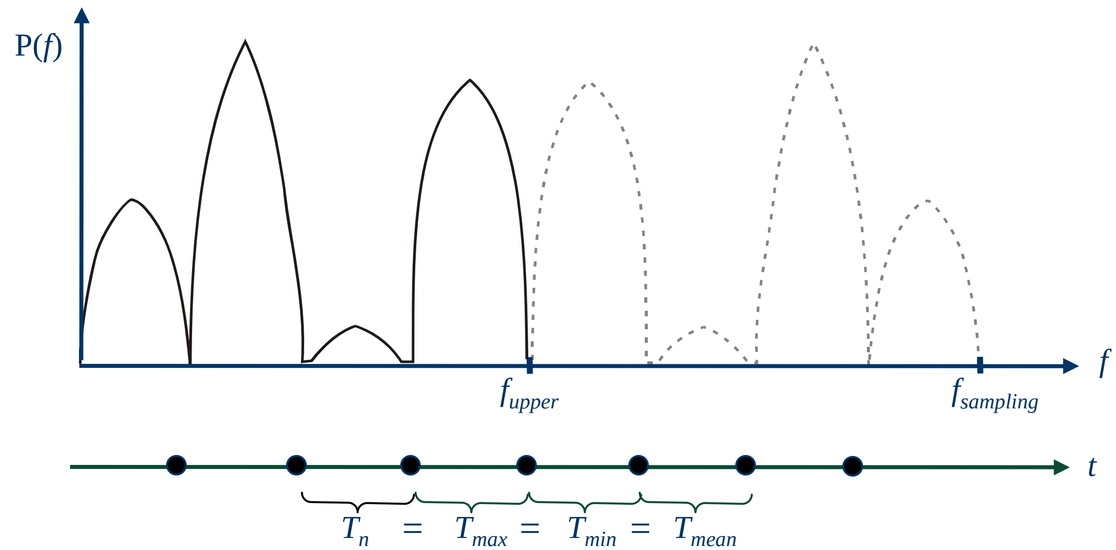
"Signal dependent sampling and reconstruction"

SIGNAL SAMPLING

- **Uniform sampling**

$$f_{sampl} = 2 f_{upper}$$

$$T = \frac{1}{2 f_{upper}}$$



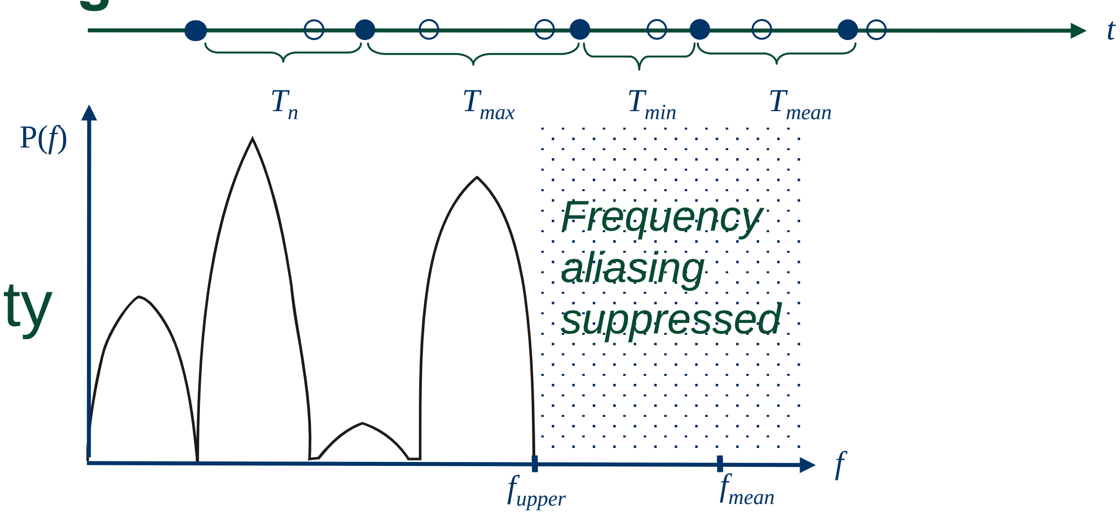
- **Nonuniform sampling**

T_{max} – maximal gap

T_{min} – minimal gap

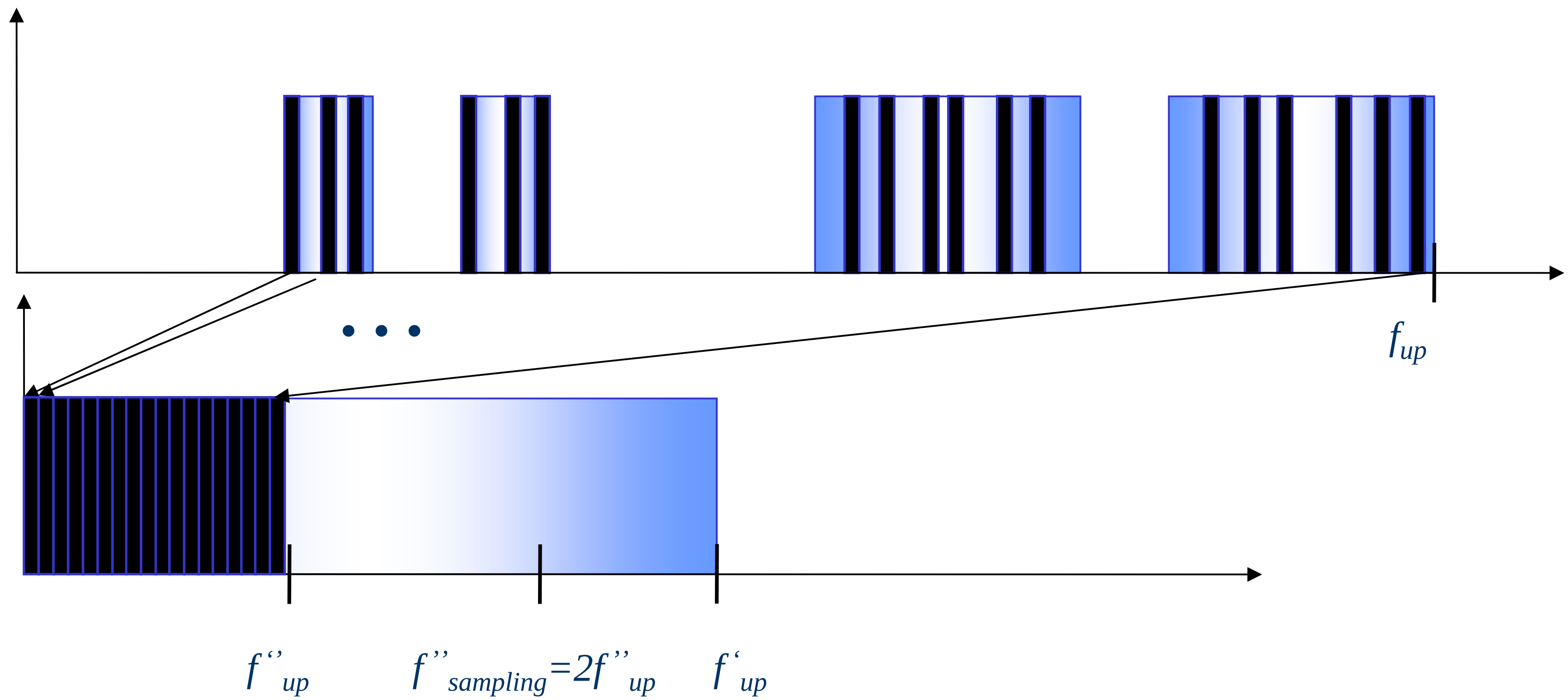
T_{mean} – sampling density

$$f_{mean} = \frac{1}{T_{mean}}$$

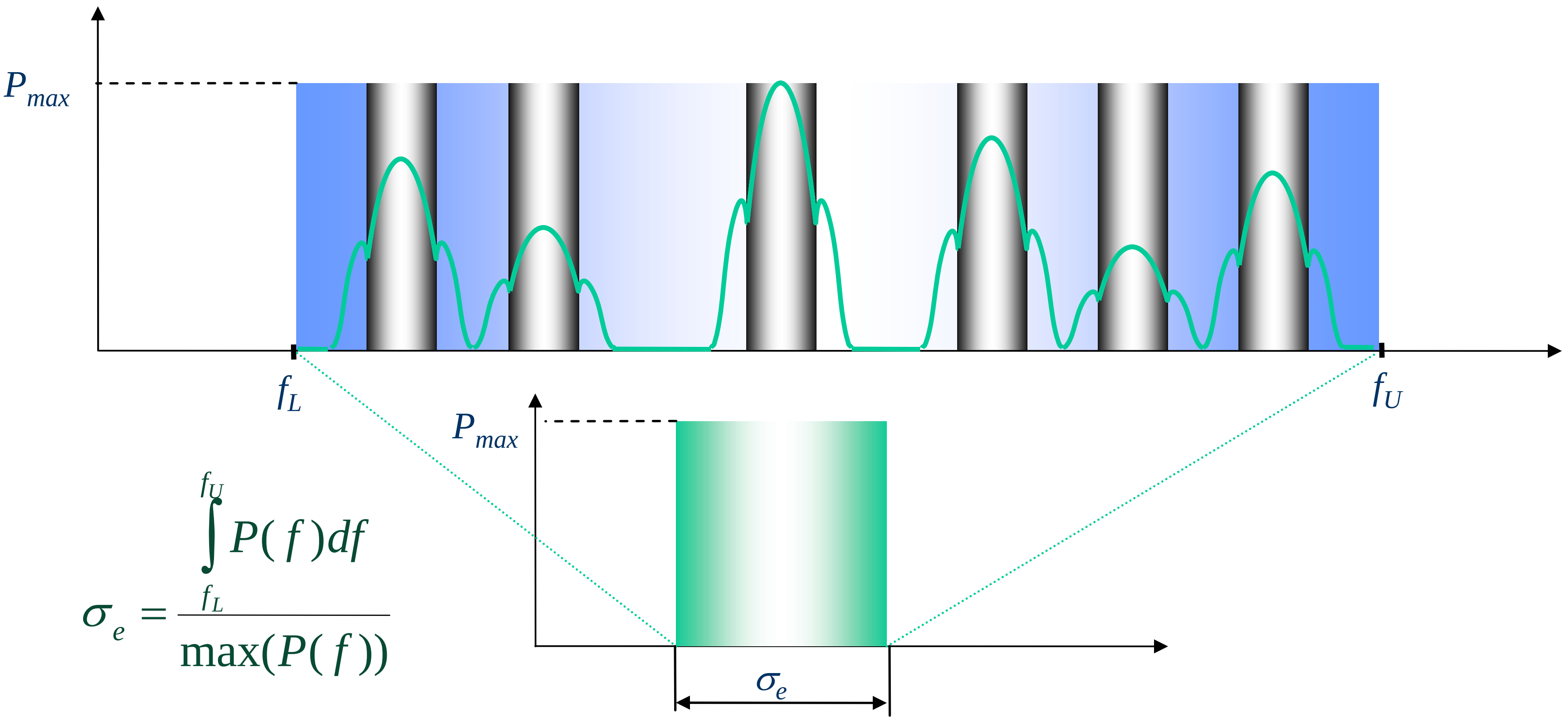


SIGNAL SAMPLING

Multi-channel multiband signal

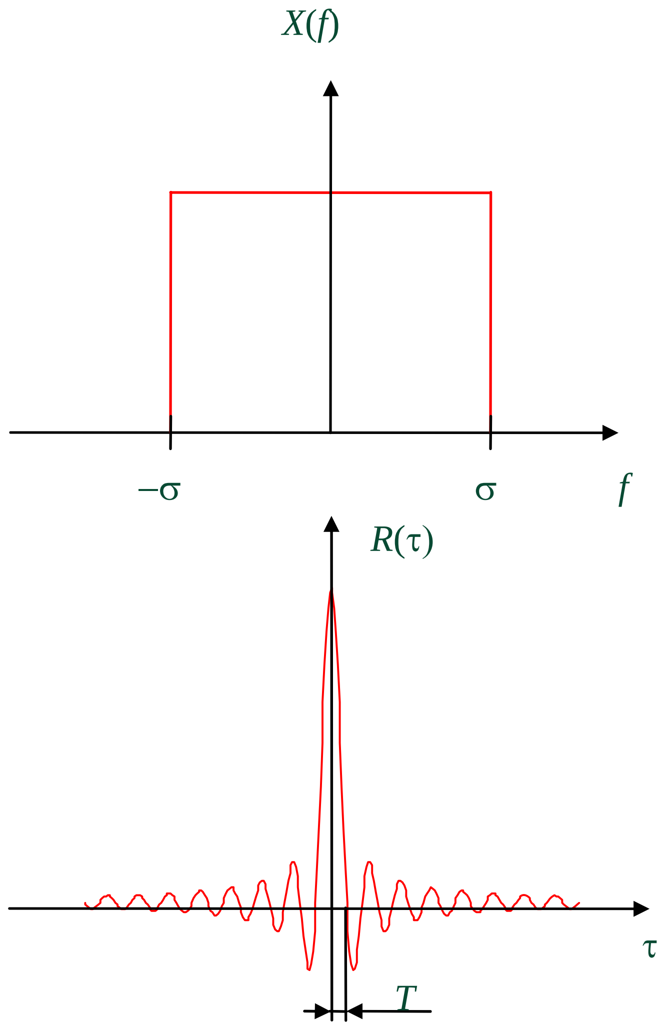


EQUIVALENT SIGNAL BANDWIDTH

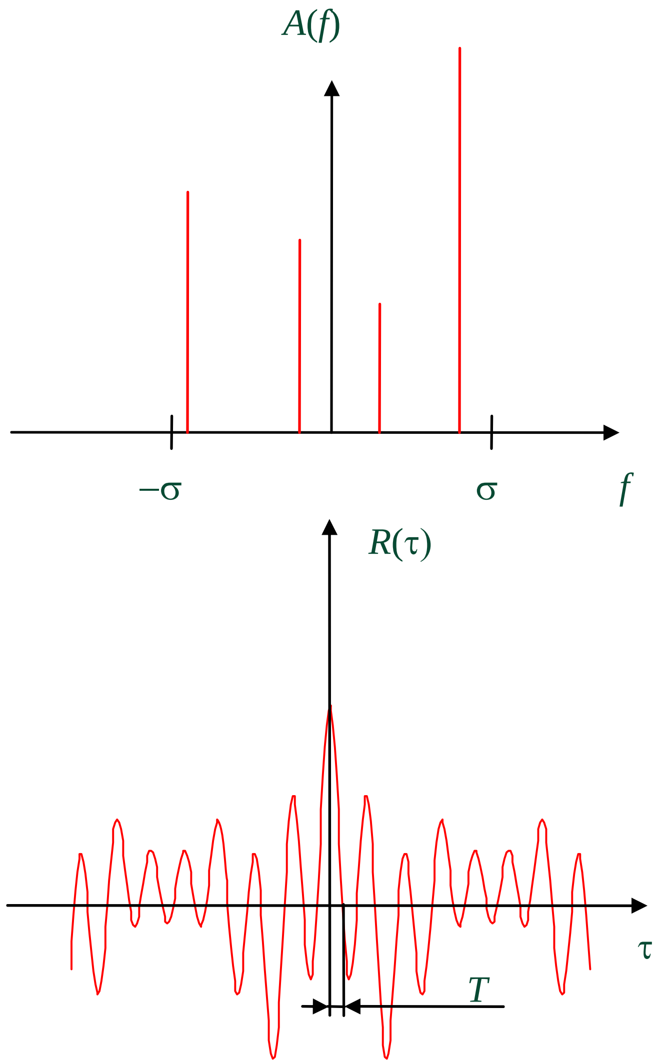


$$\sigma_e = \frac{\int_{f_L}^{f_U} P(f)df}{\max(P(f))}$$

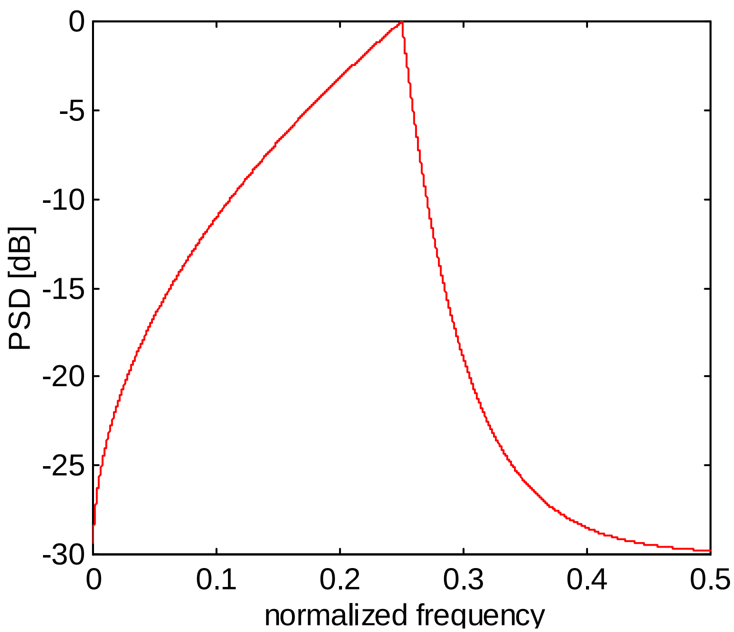
Signal with a flat PSD function

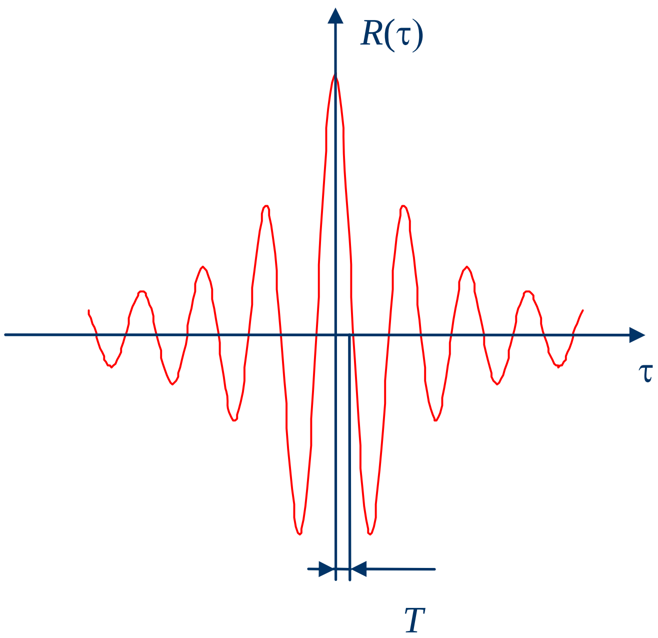


Multiple sinusoids signal



Coloured noise





$$\sigma_e = \frac{\int_{-\sigma}^{\sigma} P(f)df}{\max(P(f))}$$

$$\sigma_e \approx \tfrac{1}{3}\sigma$$

$$T_e = \frac{1}{\sigma_e}$$

PROCESSING IN TIME DOMAIN

- **Signal reconstruction:**

$$x(t)=\sum_{n=1}^{N} c_n s(t-t_n); \qquad s(t) \text{ - constructing function}$$

- **WKS theorem:**

$$x(t)=\sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}([t-nT]/T) \quad \Rightarrow \quad x(t)=\sum_{n=1}^{N} x_n \,\mathrm{sinc}(2f_u[t-t_n])$$

$$\Rightarrow \quad s(t)=\mathrm{sinc}\,2f_u t=\frac{\sin 2\pi f_u t}{2\pi f_u t}$$

- **Nonuniform sampling case:**

$$x(t)=\sum_{n=1}^{N} c_n \,\mathrm{sinc}(2f_u[t-t_n]); \qquad \mathbf{Sc}=\mathbf{x}; \qquad S_{mn}=s(t_m-t_n)$$

TIME DOMAIN ⇔ FREQUENCY DOMAIN

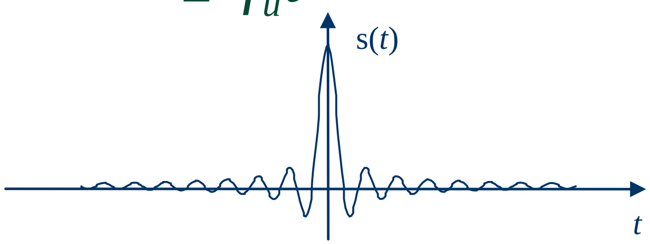
Fourier transform:

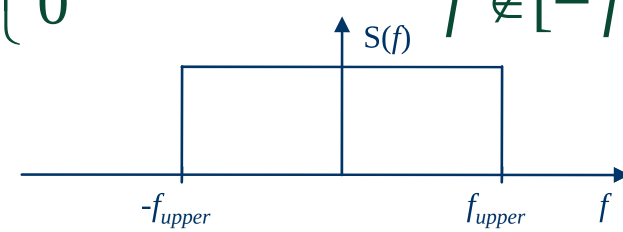
$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-i2\pi ft)dt = \mathrm{F}(x(t))$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(i2\pi ft)df = \mathrm{F}^{-1}(X(f))$$

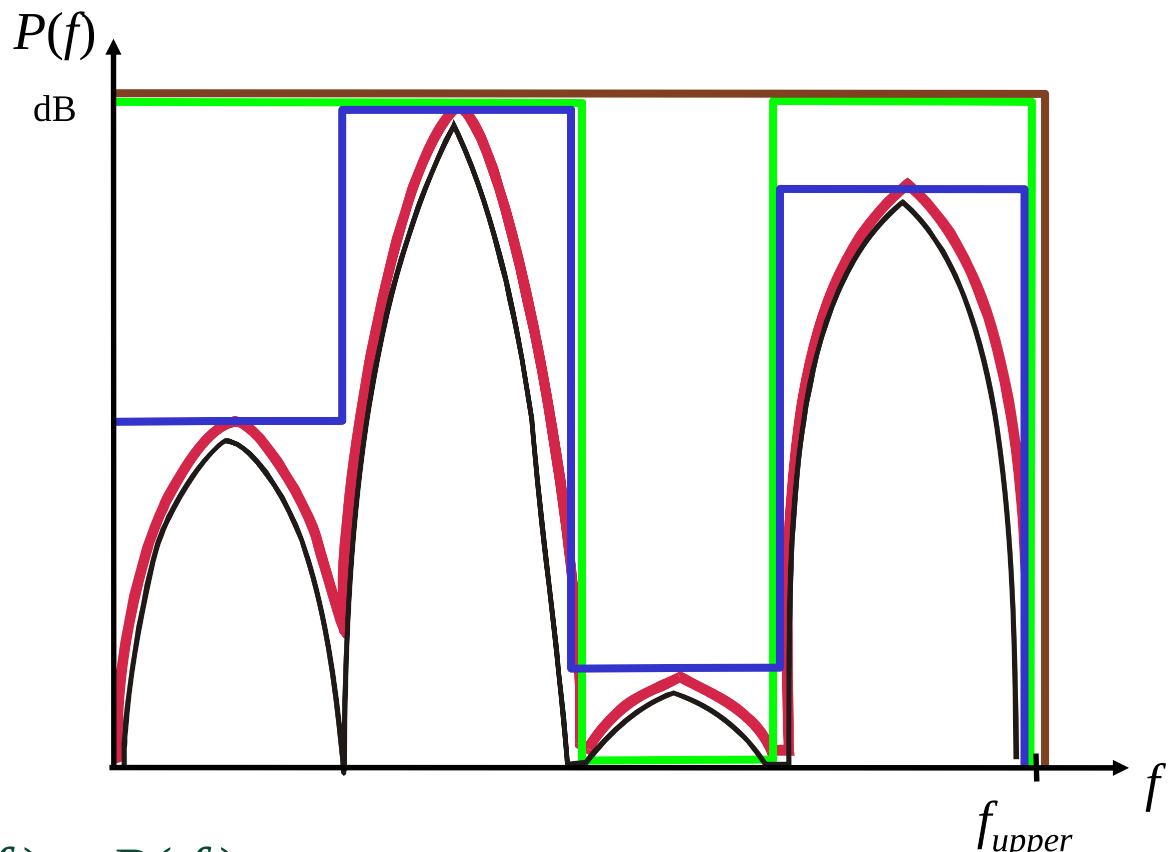
Constructing function | **Spectral support function**

$$s(t) = \frac{\sin 2\pi f_u t}{2\pi f_u t} \quad \leftrightarrow \quad S(f) = \begin{cases} \dfrac{1}{2f_{upper}} = const & f \in [-f_{upper}; f_{upper}] \\ 0 & f \notin [-f_{upper}; f_{upper}] \end{cases}$$





SPECTRAL SUPPORT FUNCTION



$$S(f) = P(f)$$

$$s(t) = \mathrm{F}^{-1}(S(f)) = r(t)$$ - autocorrelation function

ACF-based RECONSTRUCTION

- **Signal waveform reconstruction:**

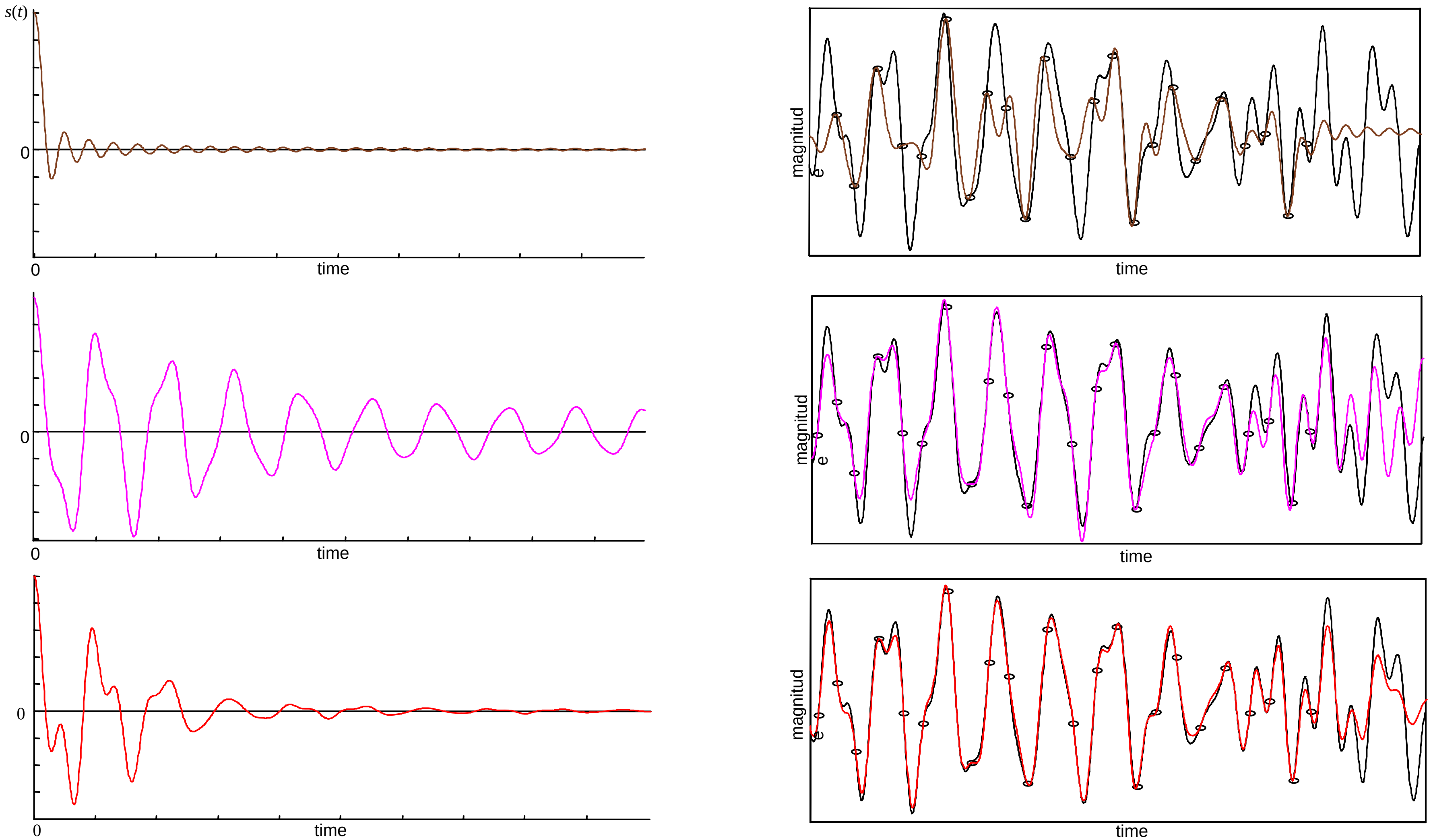
$$x_r(t) = \sum_{n=1}^{N} c_n r(t - t_n); \quad c_n \rightarrow \mathbf{Rc} = \mathbf{x}; \quad R_{mn} = r(t_m - t_n)$$

$r(t)$ - autocorrelation function $\Leftarrow$ apriori / iteratively

- **Signal spectral analysis:**

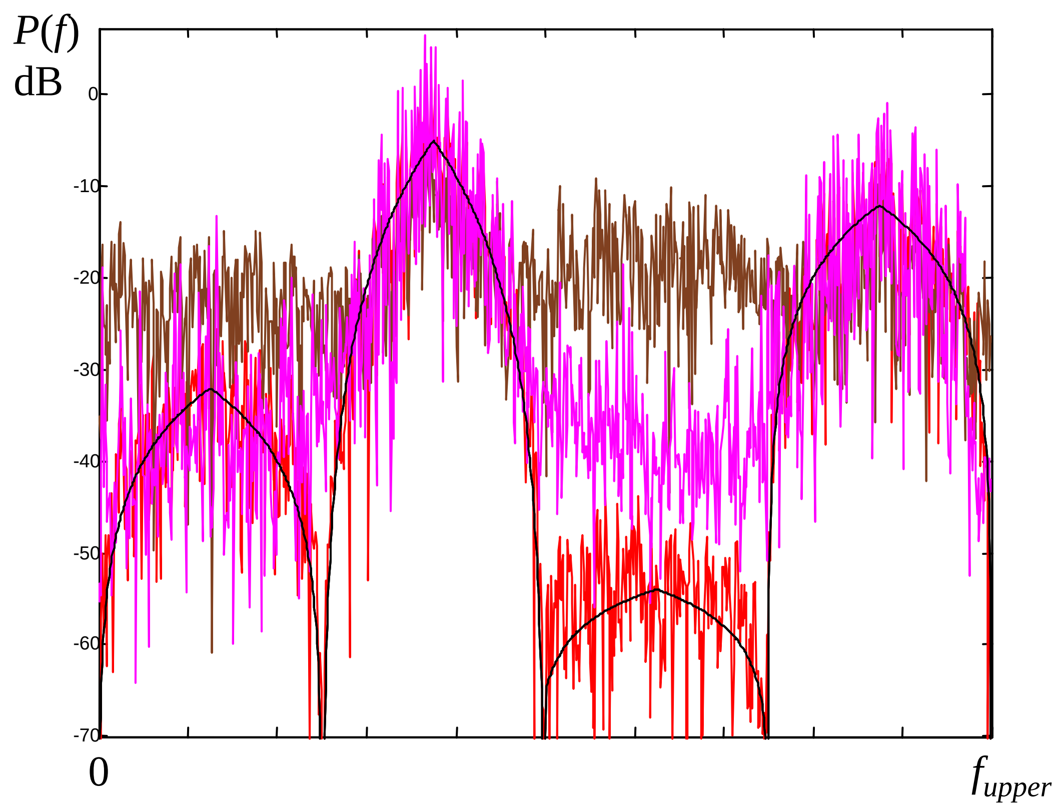
$$X_r(f) = \int_{-\infty}^{\infty} x_r(t) \exp(-i2\pi f t) dt = \int_{-\infty}^{\infty} \sum_{n=1}^{N} c_n r(t - t_n) \exp(-i2\pi f t) dt$$

EXAMPLE: WAVEFORM RECONSTRUCTION



$$N = 256; \quad T_{min} = 1.5T; \quad T_{max} = 2.5T; \quad T_{mean} = 2T$$

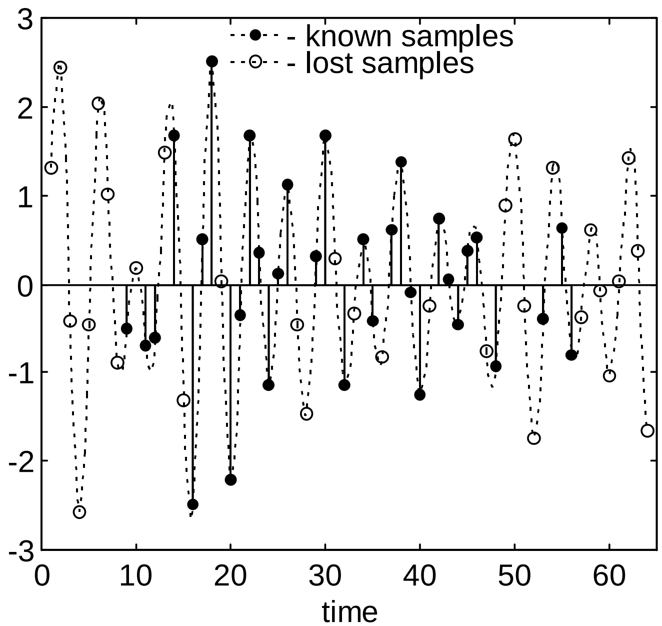
EXAMPLE: PSD FROM RECONSTRUCTED SIGNAL



$$N = 256; \quad T_{min} = 1.5T; \quad T_{max} = 2.5T; \quad T_{mean} = 2T$$

RECONSTRUCTION OF LOST SAMPLES

- Data vector **x** is obtained from band-limited to σ function $x(t)$ by uniform sampling according to Whittaker-Kotelnikov-Shannon theorem without oversampling.

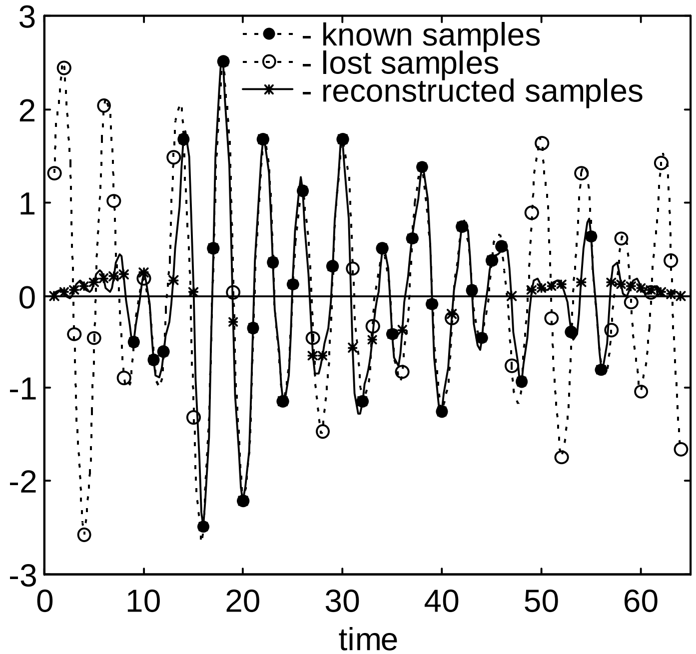


- The frame of N=64 signal samples is taken.
- The N_l=32 signal samples are lost.
- The N_k=32 signal samples are known.
- $L = \{l_1,\dots,l_{N_l}\}$ – set of subscripts of the lost samples.
- $K = \{k_1,\dots,k_{N_k}\}$ – set of subscripts of known samples.
- $\mathbf{y} = \mathbf{x}(K)$ – the vector of known samples
- The problem is to recover the lost samples $\mathbf{z} = \mathbf{x}(L)$

Minimum Norm Least Squares (MNLS) solution

$$\mathbf{z} = \mathbf{A}\mathbf{v}, \qquad A_{m,n} = \frac{\sin 2\pi(l_m - k_n)T\sigma}{\pi(l_m - k_n)T}$$

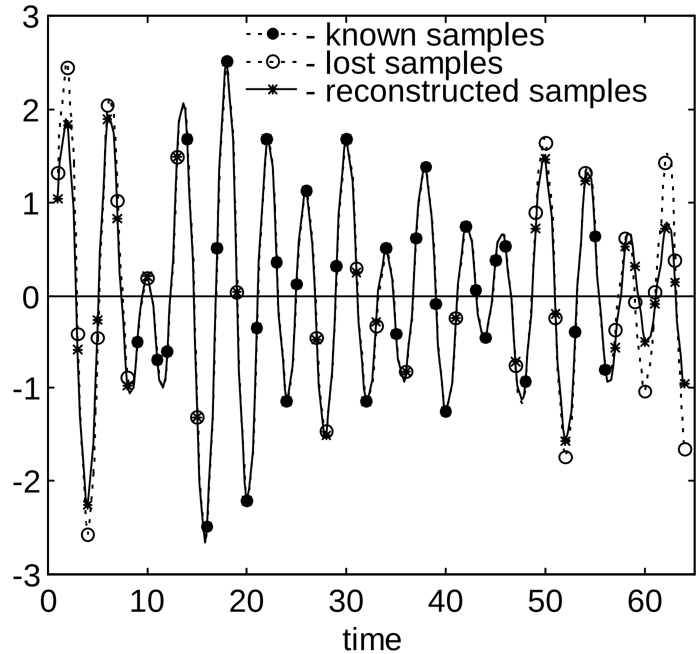
$$\mathbf{B}\mathbf{v} = \mathbf{y}, \qquad B_{m,n} = \frac{\sin 2\pi(k_m - k_n)T\sigma}{\pi(k_m - k_n)T}$$



Minimum mean-square (Wiener filter) solution

$$\mathbf{z} = \mathbf{C}\mathbf{w}, \qquad C_{mn} = R((l_m - k_n)T)$$

$$\mathbf{R}\mathbf{w} = \mathbf{y}, \qquad R_{mn} = R((k_m - k_n)T)$$

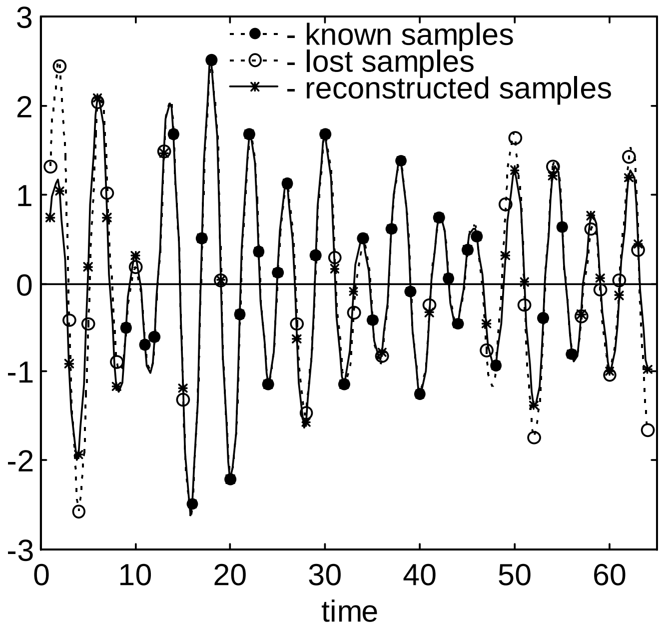


Proposed approach to recovering of lost samples

In absence of any additional information

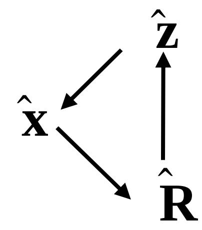
$$\hat{\mathbf{R}}^{(0)} = \left| \frac{\hat{\mathbf{x}}^{(0)}\mathbf{E}}{N} \right|^2 \mathbf{E}, \quad E_{m,n} = \exp(-j2\pi mn/N)$$

$$\hat{\mathbf{x}}^{(0)} = [\mathbf{y}\ \mathbf{z}^{(0)}], \quad \mathbf{z}^{(0)} = 0 \text{ or MNLS solution}$$



$$\begin{cases} \mathbf{z}^{(i)} = \mathbf{C}^{(i)}\mathbf{w}^{(i)} \\ \hat{\mathbf{R}}^{(i)}\mathbf{w}^{(i)} = \mathbf{y} \end{cases}$$

$$\hat{\mathbf{x}}^{(i)} = [\mathbf{y}\ \mathbf{z}^{(i)}]$$



$$\hat{\mathbf{R}}^{(i+1)} = \left| \frac{\hat{\mathbf{x}}^{(i)}\mathbf{E}}{N} \right|^2 \mathbf{E}$$

$$\Delta = \left\| \mathbf{z}^{(i+1)} - \mathbf{z}^{(i)} \right\|$$

PROCESSING IN FREQUENCY DOMAIN

- **Spectral analysis:**

$$X(f) = \sum_{n=1}^{N} x_n \, \mathrm{W}(f, t_n);$$

$\mathrm{W}(f, t_n)$ - transformation kernel

- **Fourier transform:**

$$X(f) = \sum_{n=1}^{N} x_n \exp(-i2\pi f t_n)$$

$$\mathrm{W}(f, t_n) = \{\exp(-i2\pi f t_n)\}$$

ACF-based SPECTRAL ANALYSIS

- **Spectrum:**

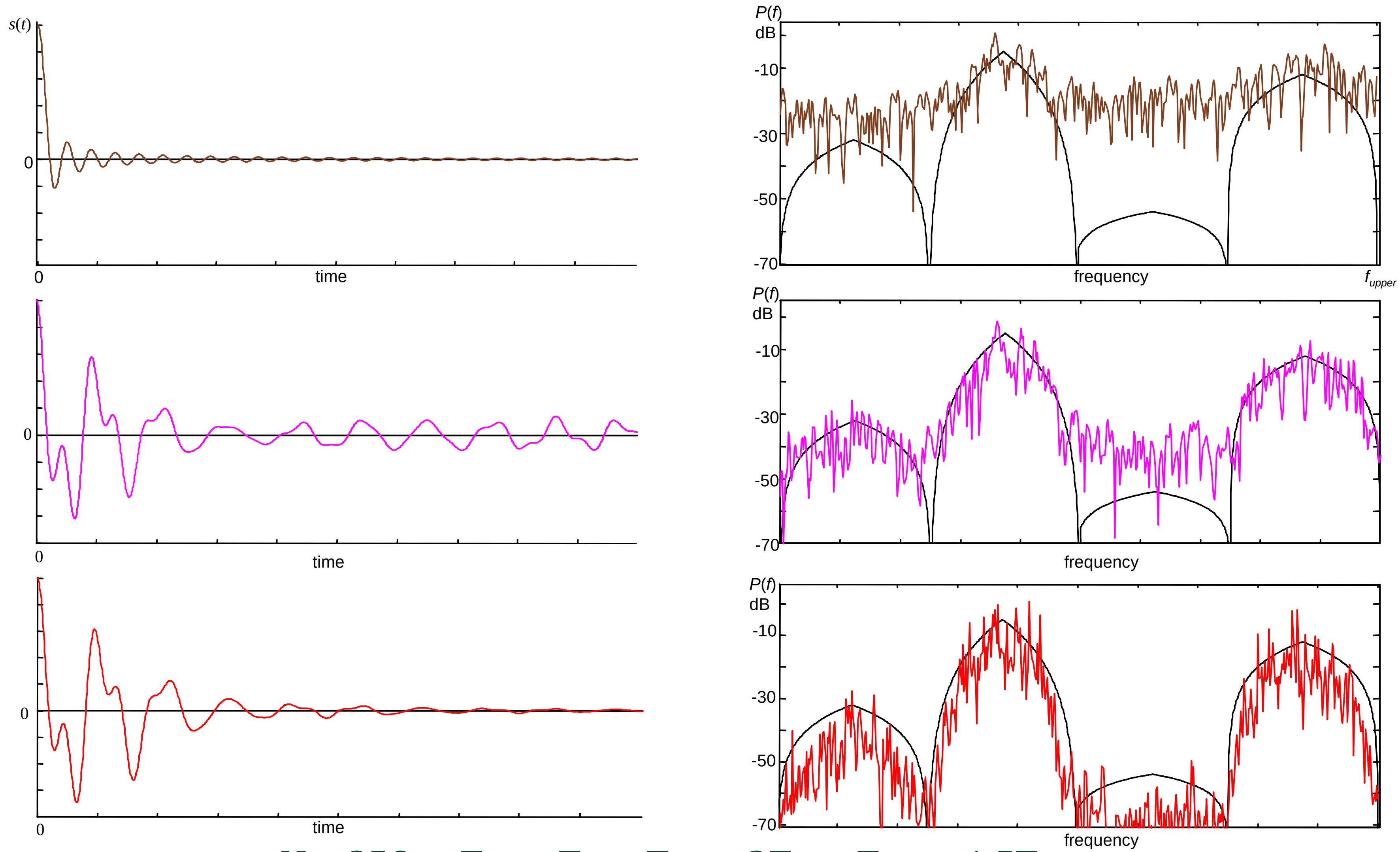
$$X_r(f) = \sum_{n=1}^{N} x_n \sum_{m=1}^{N} R_{nm}^{-1} \exp(-i2\pi f t_n);$$

$$R_{nm}^{-1} = \left(\mathbf{R}^{-1}\right)_{nm}; \quad R_{mn} = r(t_m - t_n)$$

- **Waveform:**

$$x(t) = \mathrm{F}^{-1}(X_r(f)) = \int_{-\infty}^{\infty} \left(\sum_{n=1}^{N} x_n \sum_{m=1}^{N} R_{nm}^{-1} \exp(-i2\pi f t_n) \right) \exp(i2\pi f t) df;$$

EXAMPLES: SPECTRAL ANALYSIS



$$N = 256; \quad T_{min} = T; \quad T_{max} = 2T; \quad T_{mean} = 1.5T$$

SIGNAL DEPENDENT TRANSFORM

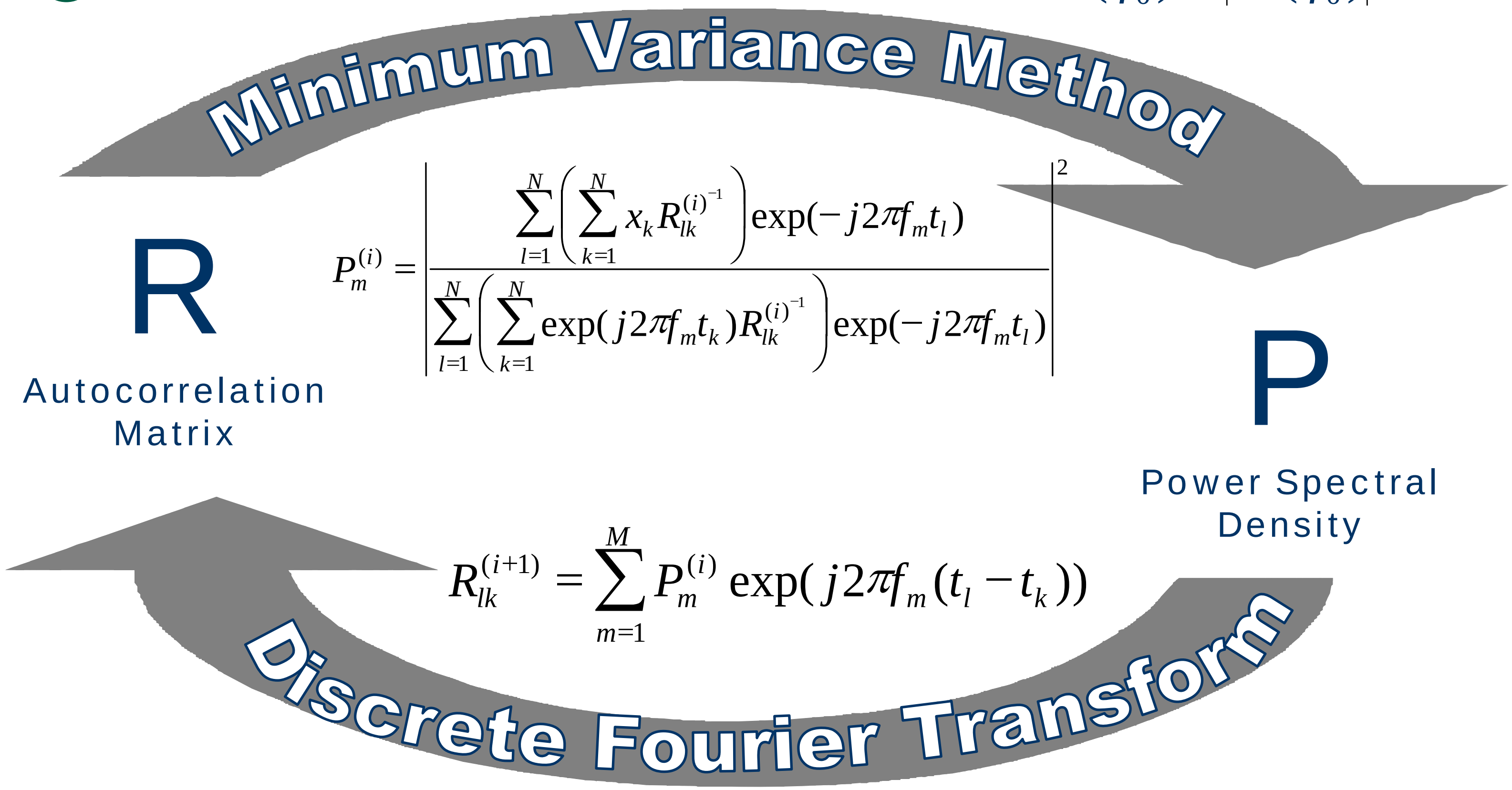
- **Narrowband filter -** $y_n = \sum_{k=1}^{K} a_k x_{n-k}$
- **Variance of the output process -** $\rho = \mathbf{a}^H \mathbf{R}\mathbf{a}$
- **Additional condition -** $\sum_{k=1}^{K} a_k \exp(-j2\pi f_0 t_k) = \mathbf{e}^H(f_0)\mathbf{a} = 1$
- **Coefficients of Minimum Variance (MV) filter -** $\mathbf{a}(f_0) = \dfrac{\mathbf{R}^{-1}\mathbf{e}(f_0)}{\mathbf{e}^H(f_0)\mathbf{R}^{-1}\mathbf{e}(f_0)}$

Features:

- **Processing at determined frequencies**
- **Complex spectral value (similarity to DFT -** $\mathbf{a}^{(F)}(f_0) = \exp(2\pi f_0 t)$**)**
- **Adaptation to the signal on each frequency (use of autocorrelation matrix)**

ITERATIVE UPDATING OF R MATRIX

- **Averaging of mutual products of samples not applicable -** $t_k - t_n \neq t_{k+i} - t_{n+i}$
- **Relation of SAF and PSD -** $R(\tau) = \int_{f_L}^{f_U} P(f)\exp(j2\pi f\tau)df$
- **Power spectral density (PSD) -** $P(f_0) = |\mathbf{xa}(f_0)|^2$



$$P_m^{(i)} = \left| \frac{\sum_{l=1}^{N}\left(\sum_{k=1}^{N} x_k R_{lk}^{(i)^{-1}}\right)\exp(-j2\pi f_m t_l)}{\sum_{l=1}^{N}\left(\sum_{k=1}^{N} \exp(j2\pi f_m t_k) R_{lk}^{(i)^{-1}}\right)\exp(-j2\pi f_m t_l)} \right|^2$$

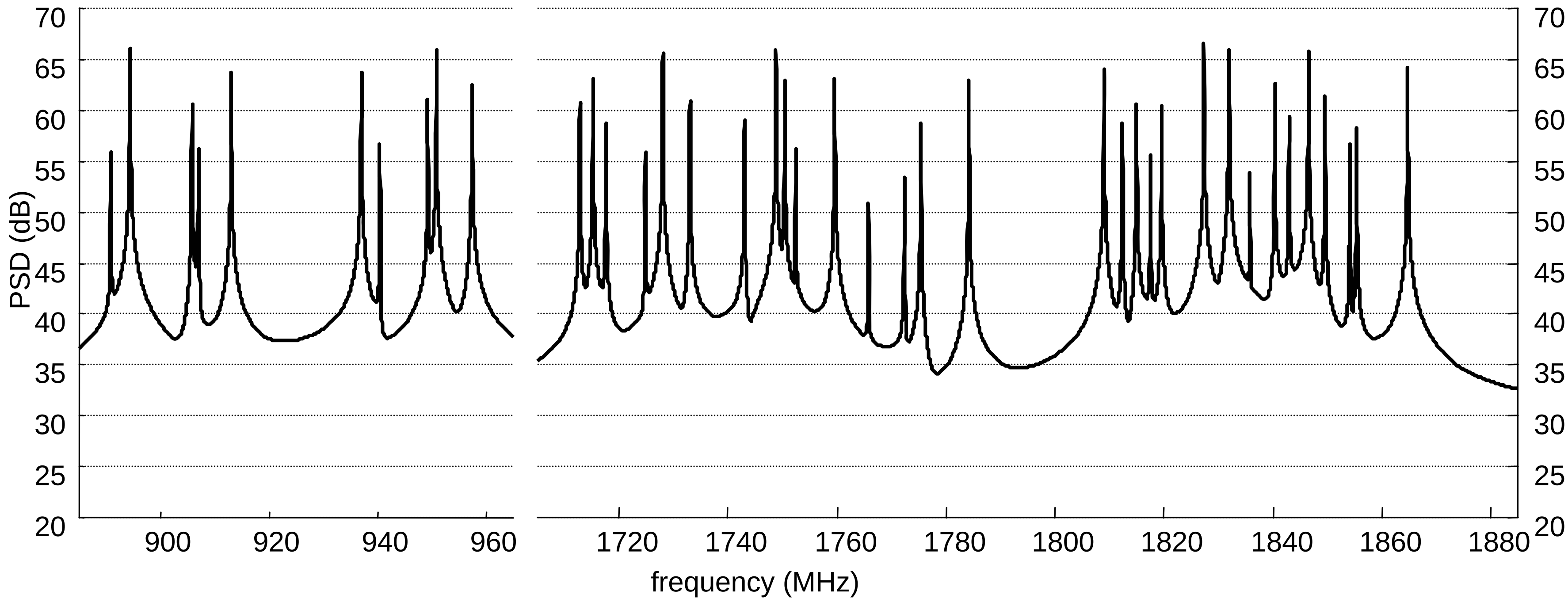
$$R_{lk}^{(i+1)} = \sum_{m=1}^{M} P_m^{(i)} \exp(j2\pi f_m (t_l - t_k))$$

GSM SIGNAL EXAMPLE

Simulation parameters:

40 frequency slots in 4 bands: 890-915 MHz, 935-960 MHz, 1710-1785 MHz, 1805-1880 MHz are used.

Total bandwidth – 200 MHz, equivalent – 20 MHz.



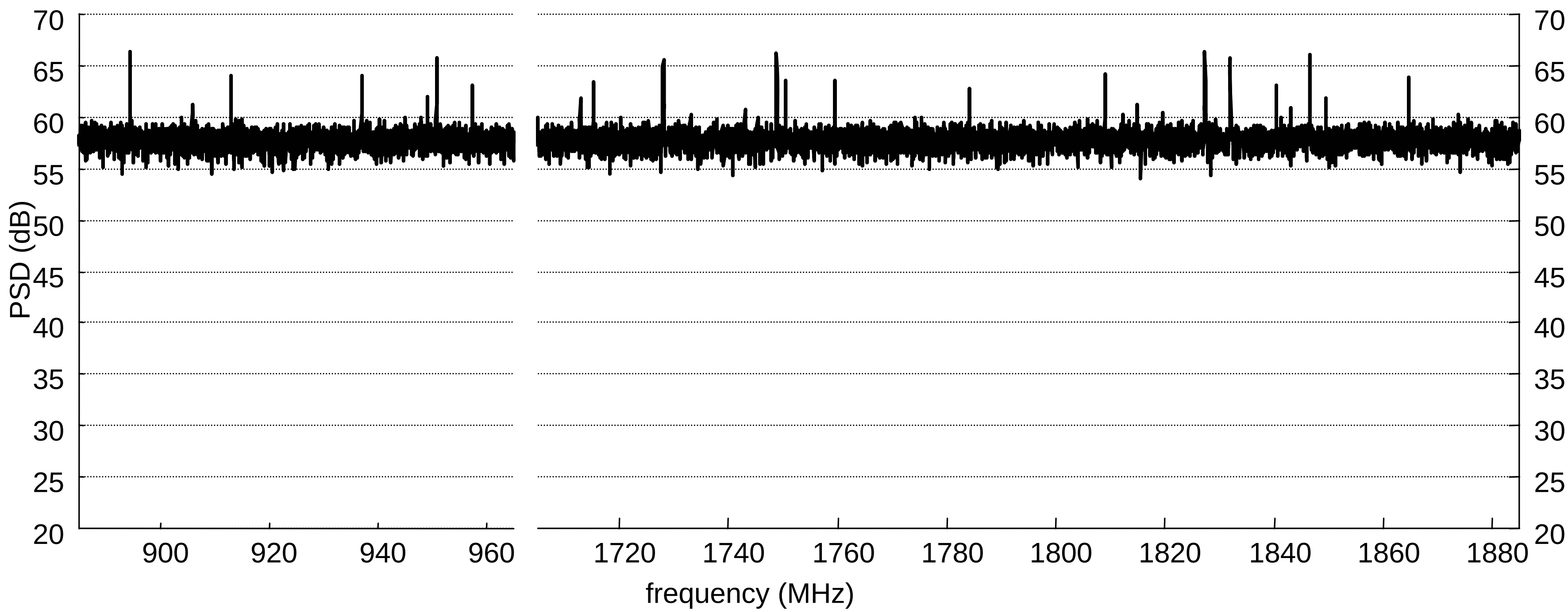
Uniform 4GHz sampling, DFT

GSM SIGNAL EXAMPLE

Simulation parameters:

40 frequency slots in 4 bands: 890-915 MHz, 935-960 MHz, 1710-1785 MHz, 1805-1880 MHz are used.

Total bandwidth – 200 MHz, equivalent – 20 MHz.



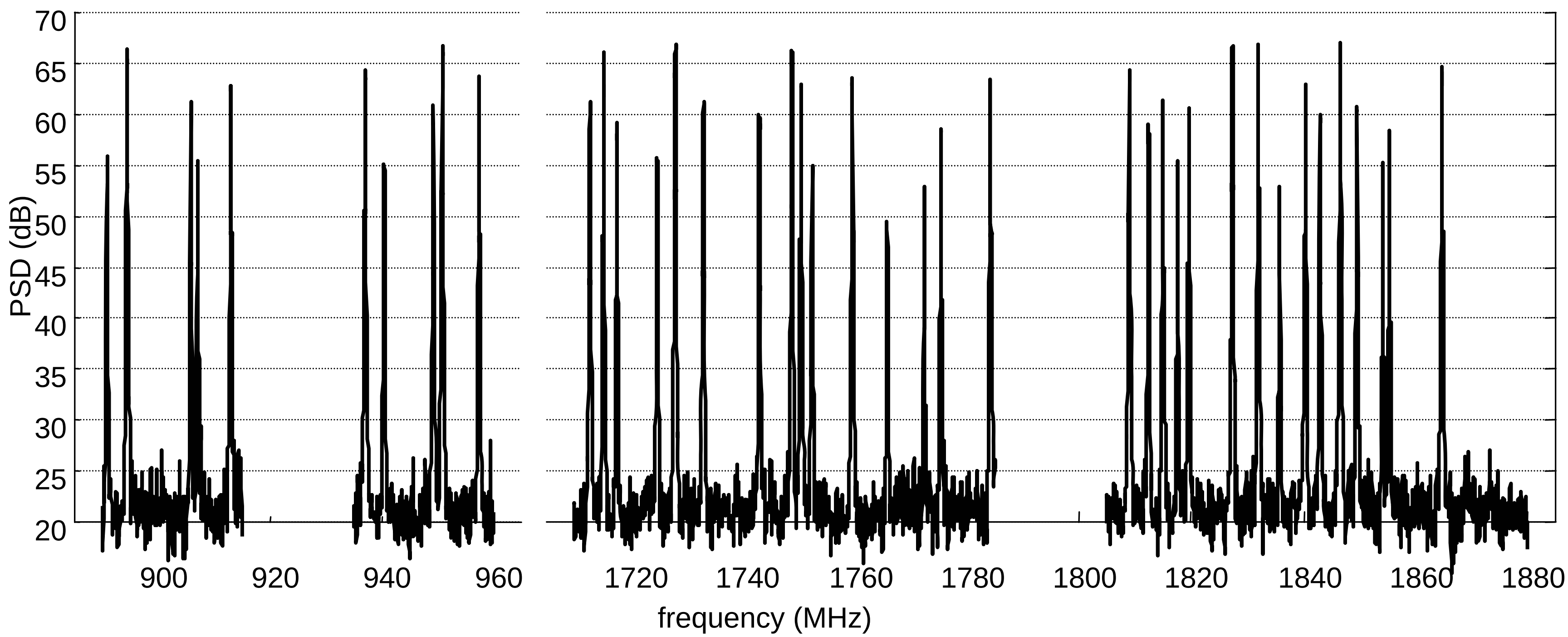
Nonuniform sampling 62.5 MSamples/s is obtained from uniform sampling 4GHz (≈ 1.56% from all samples are left), DFT

GSM SIGNAL EXAMPLE

Simulation parameters:

40 frequency slots in 4 bands: 890-915 MHz, 935-960 MHz, 1710-1785 MHz, 1805-1880 MHz are used.

Total bandwidth – 200 MHz, equivalent – 20 MHz.



Nonuniform sampling 62.5 MSamples/s is obtained from uniform sampling 4GHz (≈ 1.56% from all samples are left), iterative MV

RECONSTRUCTION RELIABILITY

Reliability in logarithmical form:

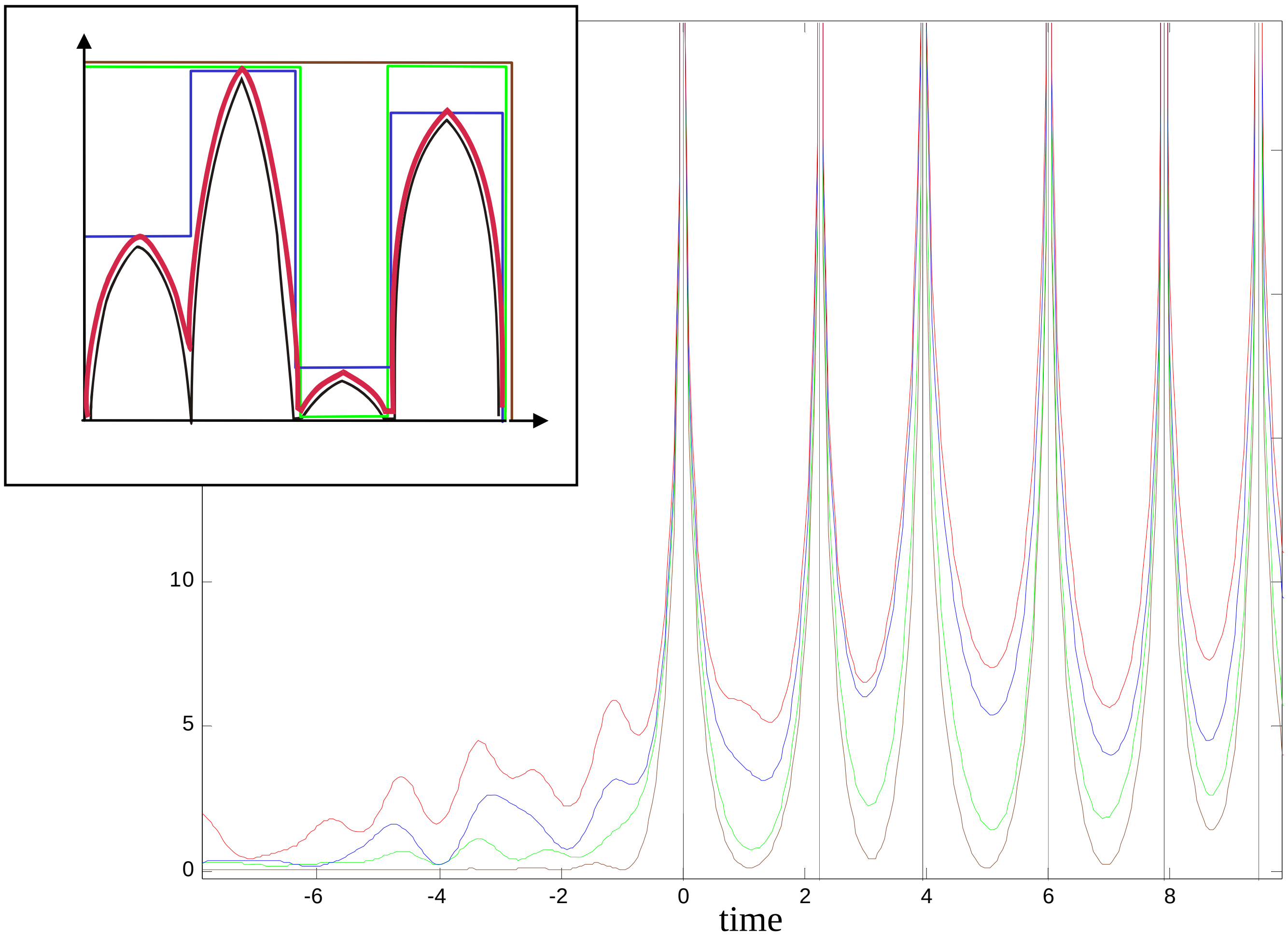
$$Q(t_0) = 10\log_{10}\left(\frac{s(0)}{s(0) - \mathbf{s}(t_0)^T \mathbf{S}^{-1}\mathbf{s}(t_0)}\right)$$

$$s(t_0)_n = s(t_0 - t_n) \qquad S_{mn} = s(t_m - t_n)$$

Reliability of SDT reconstruction:

$$Q_r(t_0) = 10\log_{10}\left(\frac{r(0)}{r(0) - \mathbf{r}(t_0)^T \mathbf{R}^{-1}\mathbf{r}(t_0)}\right)$$

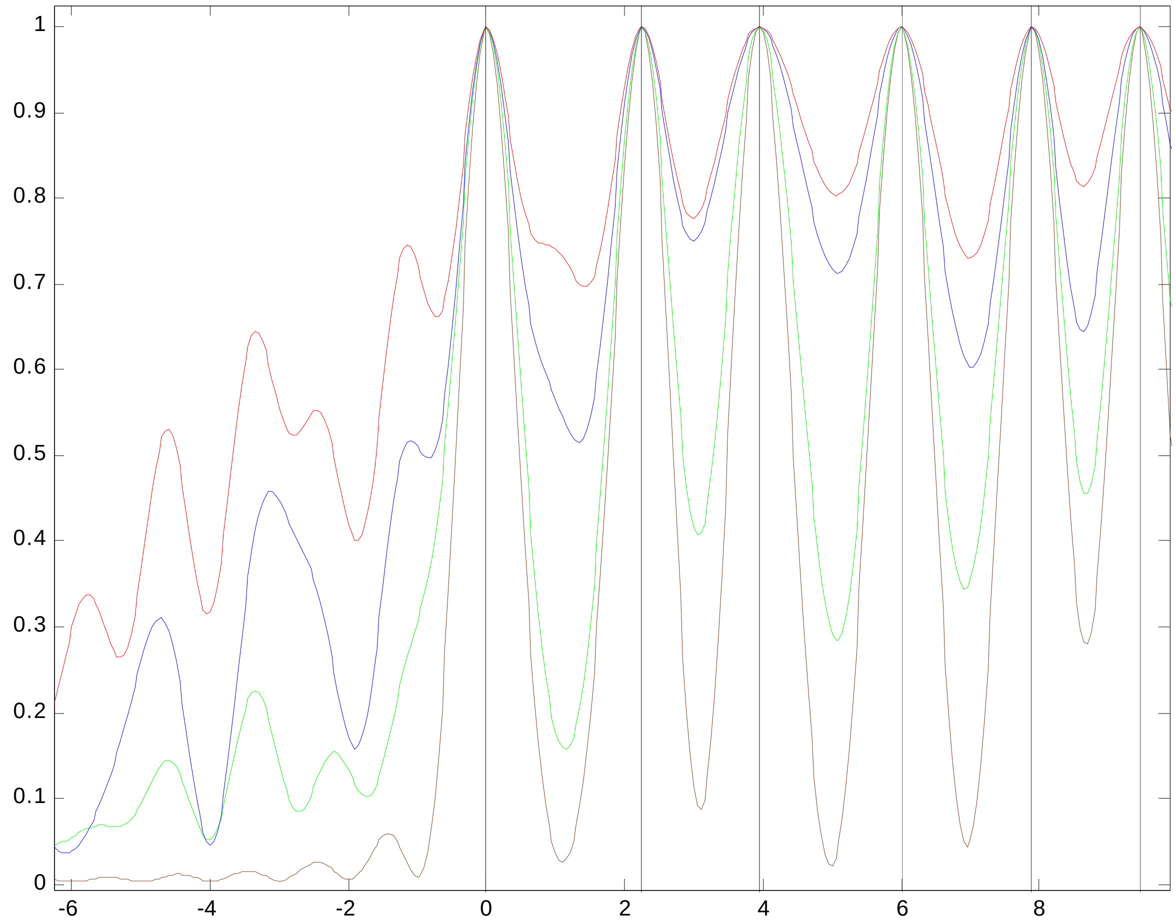
EXAMPLE OF RELIABILITY FUNCTIONS



EXAMPLE OF RELIABILITY FUNCTIONS (cont.)

$$q(t_0) = \mathbf{s}(t_0)^T \mathbf{S}^{-1} \mathbf{s}(t_0)$$

$$q_r(t_0) = \mathbf{r}(t_0)^T \mathbf{R}^{-1} \mathbf{r}(t_0)$$



ABILITIES OF PROPOSED APPROACH

- Spectral analysis
  - Complex spectral values (amplitudes and phases)
  - PSD function values
- Correlation analysis
- Signal reconstruction
  - Inverse DFT from complex spectrum
- Filtering

ADVANTAGES OF PROPOSED APPROACH

- Applicability to various kinds of samplings (deals with less samples than required by Nyquist)
- Sampling independent from actual frequency boundaries of signal
- Limitation of side-loop effect
- Limitation of noise floor from nonuniform sampling

CONCLUSIONS

- Uniform sampling can be characterized by one value – sampling period T, while nonuniform sampling have to be characterized by several parameters

- Significant is maximal interval between two consequent samples T_{max}

- Typically PSD function of band-limited signal is characterized by one value – bandwidth, while signal spectral characteristics can differ substantially

- Traditional DSP methods support whole frequency range of analysis equally

- Presented approach is based on an idea **not to support whole frequency range equally, but to concentrate processing on the powerful components**

- Autocorrelation function is used as constructing function for signal reconstruction $\Leftrightarrow$ Power spectral density function is used as spectral support function

- Information about autocorrelation function can be known “a priori” or obtained in an iterative way

- Reliability function demonstrates the eventual enhancement of signal processing results

- In combination with nonuniform sampling (to suppress frequency aliasing) proposed approach enhances signal processing in time and frequency domains also for case if $T_{max} \geq T$

DISCUSSIONS

- Characterization of signal bandwidth and autocorrelation function
- Parameters of nonuniform sampling
- Convergence of iterations
- Reducing of required mathematical calculations